EXTREME POINTS OF THE CLASS OF DISCRETE
DECREASING FAILURE RATE AVERAGE LIFE DISTRIBUTIONS

by

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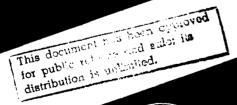James Lynch[3], and Frank Proschan[4]

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Extreme Points of the Class of Discrete Decreasing Failure Rate
Average Life Distributions

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ABSTRACT

We show that the class of discrete decreasing failure rate average
(discrete DFRA) life distributions is a convex set. We then obtain the
extreme points of this class. Finally we show how to represent any discrete
DFRA life distribution as a mixture of these extreme points.

Key Words: Discrete decreasing failure rate average, extreme points, convex
class, reliability, life distribution, representation, mixture of distributions.

1. Introduction and Summary.

Distributions with decreasing failure rate (DFR) occur frequently in reliability theory and application. See, for example, Barlow and Proschan (1975) and Proschan (1963). Less common is the use of distributions with decreasing failure rate average (DFRA). However, it is easy to formulate models and find real life examples of DFRA distributions. We list a few, since the literature seems devoid of such cases.

(1) The life length of a retail outfit merchandising seasonal goods may be DFRA. The failure rate tends to decrease with increased experience, growing capital, public recognition of the product or company name, and other factors monotonic with age. However, the seasonal factor prevents the failure rate from being monotonically decreasing, so that only the average failure rate is decreasing.

(2) A device operates 16 hours a day, say. On the ith day, the failure rate during operation is λ_i and is 0 during the 8 hours of "rest". Suppose $\lambda_1 > \lambda_2 > \lambda_3 > \ldots$, but, in addition, are such that the cumulative failure rate H(t) over time (including rest periods) satisfies: $\frac{1}{t}H(t)$ is decreasing. The graph in Fig. 1 displays the situation.

Note that $\frac{1}{t}H(t)$ is decreasing, but H(t) is not concave. Thus the underlying distribution is DFRA but not DFR.

Other practical examples may be listed in the continuous case. In a similar fashion, DFRA life distributions occur in the discrete case, in which age is measured by the number of cycles that have occurred since the unit was initially put into operation.

Fig. 1. Cumulative hazard function of a DFRA distribution.

In this paper, we obtain the extreme points of the convex class of discrete DFRA distributions. In addition, we show constructively how to represent any discrete DFRA distribution as a convex combination of extreme points.

As is well known in optimization theory, from a knowledge of the extreme points it may be possible to obtain maxima or minima of certain functionals of discrete DFRA distributions. Bounds and inequalities for discrete DFRA distributions may also be derivable from a knowledge of the extreme points of the discrete DFRA class.

2. Preliminaries.

A distribution F is a discrete life distribution if its support is contained in the set $\{0, 1, \ldots\}$. We denote the corresponding survival function $1 - F$ by $\overline{F}$. We define the hazard function $h(F, x)$ of a discrete life distribution as $-\ln\overline{F}(k - 1)$ for $x \in [k, k + 1)$ and $k = 0, 1, \ldots$.

We define two concepts used in the sequel.

Definition 2.1. Let G be a class of distribution functions. Then G is a convex class if $F \equiv \theta F_1 + (1 - \theta)F_2 \in G$ whenever $F_1, F_2 \in G$ and $\theta \in [0, 1]$.

Definition 2.2. Let G be a convex class of distribution functions, and let $F \in G$. Then F is an extreme point of G if there are no two distinct distribution functions $F_1, F_2 \in G$ and a real number $\theta \in (0, 1)$ such that $F \equiv \theta F_1 + (1 - \theta)F_2$.

Next we present the class of discrete life distributions that is the subject of our analysis.

Definition 2.3. Let F be a discrete life distribution. Then F is decreasing failure rate average (DFRA) if $x^{-1}h(F, x)$ is nonincreasing in $x \in (0, \infty)$.

We denote by G_D the class of discrete DFRA life distributions. Throughout we define $(\overline{F}(-1))^{0-1} \equiv 0$.

3. The Extreme Points of the Discrete DFRA Class.

In this section we identify the extreme points of the class of discrete DFRA life distributions. The function $g(\theta, x, y) = -\ell n\{\theta e^{-x} + (1 - \theta)e^{-y}\}$, $\theta \in [0, 1]$, $x, y \in [0, \infty)$, plays a key role in our analysis. First, we prove three properties of $g(\theta, x, y)$ used in the sequel.

Lemma 3.1. Let $\theta \in (0, 1)$. Then (i) $g(\theta, x, y)$ is strictly increasing in x and y, (ii) For $\alpha \in [0, 1]$, and $x, y \in [0, \infty)$, $g(\theta, \alpha x, \alpha y) \geq \alpha g(\theta, x, y)$, and (iii) For $\alpha \in (0, 1)$ and $x, y \in [0, \infty)$, $g(\theta, \alpha x, \alpha y) = \alpha g(\theta, x, y)$ iff $x = y$.

<u>Proof</u>. (i) follows in a straightforward way.

(ii) and (iii). Define the random variable Z as follows:

$$
Z = \begin{cases} e^{-x} & \text{with probability } \theta \\ e^{-y} & \text{with probability } 1 - \theta \end{cases}
$$

Then $g(\theta, x, y) = -\ell n EZ$, and $g(\theta, \alpha x, \alpha y) = -\ell n EZ^{\alpha}$. Consequently, (ii) and (iii) follow by the Liapounov Inequality [Chung (1974), p. 47]. ||

Next we show that the class G_D is convex.

Lemma 3.2. The class of discrete DFRA life distributions is convex.

<u>Proof</u>. Let $F \equiv \theta F_1 + (1 - \theta)F_2$, where $F_1, F_2 \in G_D$ and $\theta \in (0, 1)$. We show that $F \in G_D$.

Let $k \in \{1, 2, \ldots\}$. Then $H(F, k) = g\{\theta, H(F_1, k), H(F_2, k)\}$. By Lemma 3.1(i):

$$H(F, k) \geq g\{\theta, k(k + 1)^{-1}H(F_1, k + 1), k(k + 1)^{-1}H(F_2, k + 1)\}.$$

By Lemma 3.1(ii):

$$g\{\theta, k(k + 1)^{-1}H(F_1, k + 1), k(k + 1)^{-1}H(F_2, k + 2)\} \geq$$

$$k(k + 1)^{-1}g\{\theta, H(F_1, k + 1), H(F_2, k + 1)\} = k(k + 1)^{-1}H(F, k + 1).$$

Consequently the desired result follows. ||

Define $\Delta(F, 0) = -\infty$, and $\Delta(F, k) = (k + 1)^{-1}H(F, k + 1) - k^{-1}h(F, k)$, $k = 1, 2, \ldots$. To accomplish the objective of this section we need the following lemma.

Lemma 3.3. Let $F \equiv \theta F_1 + (1 - \theta)F_2$, where F_1, $F_2 \in G_D$ and $\theta \in (0, 1)$. Assume $\Delta(F, k) = 0$ for some positive integer k. Then $h(F_1, k) = h(F_2, k) = h(F, k)$, and $h(F_1, k + 1) = h(F_2, k + 1) = h(F, k + 1)$.

Proof. First by Lemma 3.1(i) and (ii):

$$k^{-1}H(F, k) = k^{-1}g\{\theta, H(F_1, k), H(F_2, k)\}$$

$$\geq k^{-1}g\{\theta, k(k + 1)^{-1}h(F_1, k + 1), k(k + 1)^{-1}H(F_2, k + 1)\}$$

$$\geq (k + 1)^{-1}g\{\theta, H(F_1, k + 1), h(F_2, k + 1)\}$$

$$= (k + 1)^{-1}H(F, k + 1).$$

Since $\Delta(F, k) = 0$, the extreme values in the preceding chain of inequalities are equal. Thus, by Lemma 3.1(i), $\Delta(F_j, k) = 0$, $j = 1, 2$, and by Lemma 3.1(iii), $h(F_1, k + 1) = H(F_2, k + 1)$. Consequently the desired results follow. ||

To describe the extreme points of G_D we need the following definition and notation.

Definition 3.4. Let F be a discrete life distribution and $k_1 < k_2$ be two integers in the support of F. Then k_1, k_2 are <u>successive support points</u> if no integer in the interval (k_1, k_2) belongs to the support of F.

Let $G_{D,e} = \{F: F \in G_D$, and for every two successive support points of F, k_1, k_2, $\Delta(F, k_1) = 0$ or $\Delta(F, k_2) = 0\}$.

We are ready now to identify the extreme points of G_D.

Theorem 3.5. $G_{D,e}$ is the class of all extreme points in the class of discrete DFRA life distributions.

Proof. First, we show that all the life distributions in $G_{D,e}$ are extreme points. Let $F \equiv \theta F_1 + (1 - \theta)F_2$, where F_1, $F_2 \in G_D$, $F \in G_{D,e}$, and $\theta \in (0, 1)$. We show that $F_1 \equiv F_2$.

Let $d = \text{Sup}\{q: H(F_1, j) = H(F_2, j), j = 0, \ldots, q\}$. To prove that $F_1 \equiv F_2$, it suffices to show that $d = \infty$. Assume $d < \infty$. Then there is a positive integer k such that $d < k$, and d, k are two successive support points of F. By the definition of d and Lemma 3.3, $\Delta(F, d) < 0$. Thus, $\angle(F, k) = 0$. Consequently by Lemma 3.3, $d \geq k + 1 > d$, a contradiction. Hence $d = \infty$ and F is an extreme point in the class G_D.

To complete the proof of the theorem, we show that a discrete DFRA life distribution that does not belong to $G_{D,e}$ is not an extreme point in G_D. To show that a discrete DFRA life distribution is not an extreme point, it suffices to prove that the life distribution can be written as a proper convex combination of two distinct discrete DFRA life distributions. Let $F \in G_D$ such that $F \notin G_{D,e}$. We show that F is not an extreme point.

Let k_1, k_2 be two successive support points of F, such that $k_2 > k_1 \geq 0$, and $\Delta(F, k_j) < 0$, $j = 1, 2$. Let $F_1(k) = F_2(k) = F(k)$ for $k \notin (k_1 - 1, k_2 - 1)$, $F_1(k) = \max\{[F(k_1 - 1)]^{1+k_1^{-1}}, F(k_2)\}$, $F_2(k) = \min\{[F(k_2)]^{k_2/(k_2+1)}, F(k_1 - 1)\}$ for $k \in (k_1 - 1, k_2 - 1]$ and let $\theta = [F(k_1) - F_1(k_1)][F_2(k_1) - F_1(k_1)]^{-1}$. For the sake of clarity we recall that $[F(-1)]^{0^{-1}}$ is defined as zero. Finally, observe that F_1, F_2 are two distinct DFRA life distributions, that $\theta \in (0, 1)$, and that $F \equiv \theta F_1 + (1 - \theta)F_2$. Consequently F is not an extreme point in G_D. ||

4. Representation of a Discrete DFRA Life Distribution as a

Mixture of Extreme Points.

In this section we prove that every discrete DFRA life distribution can be presented as a mixture of the extreme points of the discrete DFRA class. More explicitly, for any discrete DFRA life distribution F, we first construct a probability space $(\Omega_F, \, \mathcal{B}_F, \, P_F)$. We then define for each $\omega \in \Omega_F$ a discrete life distribution $G_F(\cdot, \, \omega)$ that belongs to the class of the discrete DFRA extreme points. Finally, we prove that:

(4.1) $\qquad F(k) = \int_{\Omega_F} G(k, \, \omega) dP_F(\omega), \; k = 0, 1, \, \dots \, .$

Let $F \in G_{D,e}$. Define $\Omega_F = \{1\}$, $\mathcal{B}_F = \{\emptyset, \{1\}\}$, $P_F\{1\} = 1$, and $G(\cdot, \{1\}) = F(\cdot)$. Then clearly $F(k) = \int_{\Omega_F} G(k, \, \omega) dP_F(\omega), \; k = 0, 1, \, \dots \, .$ Thus, to prove Statement (4.1) it suffices to consider discrete DFRA life distributions that are not extreme points.

Let F belong to G_D but not to $G_{D,e}$. Next, we construct the probability space $(\Omega_F, \, \mathcal{B}_F, \, P_F)$, define the extreme discrete DFRA life distributions $G(\cdot, \, \omega)$, $\omega \in \Omega_F$, and prove Statement (4.1). Let m be the number (possibly infinite) of all pairs of nonnegative integers $(k_q, \, k_{q+1})$, $1 \le q < 2m + 1$, such that:

(4.2) $\qquad k_q, \; 1 \le q < 2m + 1$, is a strictly increasing sequence,

(4.3) $\qquad \Delta(F, \, k_q) < 0, \; 1 \le q < 2m + 1$, and

(4.4) $\qquad k_{2q-1}, \; k_{2q}$ are successive support points of F for $1 \le q < m + 1$.

Further, let $I_{2q} = \{k. \ k \in (k_{2q-1} - 1, k_{2q} - 1 \}, \ 1 \le q < m + 1,$

$$(4.5) \qquad \bar{F}_1(k) = \begin{cases} \max\{ \_\bar{F}(k_{2q-1} - 1)_\lrcorner^{1+k_{2q-1}^{-1}}, \ \bar{F}(k_{2q})\}, & k \in I_{2q}, \ 1 \le q < m + 1 \\ \\ F(k) & , \quad k \notin I_{2q}, \ 1 \le q < m + 1 \end{cases}$$

$$(4.6) \qquad \bar{F}_2(k) = \begin{cases} \min\{ \bar{F}(k_{2q})_\lrcorner^{k_{2q}/(k_{2q}+1)}, \ F(k_{2q-1} - 1)\}, & k \in {}_{2q}, \ 1 \le q < m + 1 \\ \\ F(k) & , \quad k \notin I_{2q}, \ 1 \le q < m + 1, \end{cases}$$

and

$$(4.7) \qquad \theta_{2q} = \lceil \bar{F}(k_{2q}-1) - \bar{F}_1(k_{2q}-1) \rfloor \lceil \bar{F}_2(k_{2q}-1) - \bar{F}_1(k_{2q}-1) \rfloor^{-1}, \ 1 \le q < m + 1.$$

For the sake of clarity, note that F_1, $F_2 \in G_{D,e}$, that $\theta_{2q} \in (0, 1)$, $1 \le q < m + 1$, and that for $k \in I_{2q}$, $1 \le q < m + 1$,

$$(4.8) \qquad \bar{F}(k) = \theta_{2q}\bar{F}_2(k) + (1 - \theta_{2q})\bar{F}_1(k).$$

We now construct the probability space (Ω_F, β_F, P_F). Let

$$\Omega_F = \{\omega: \ \omega = (\omega_{2q}, \ 1 \le q < m + 1), \ \omega_{2q} \in \{0, 1\}, \ 1 \le q < m + 1\},$$

β_F = the set of all subsets of Ω_F, and let

$$P_F\{\omega: \ \omega_{2q} = \delta_q, \ q = 1, \ldots, k\} = \prod_{q=1}^{k} \theta_{2q}^{\delta_q}(1 - \theta_{2q})^{1-\delta_q},$$

where $1 \le k < m + 1$, and $\delta_1, \ldots, \delta_k \in \{0, 1\}$. For the case $m = \infty$, we extend

P_F to all subsets of Ω_F by the Caratheodory Extension Theorem ialmos (1965), p. 54 .

Next, we define for each $\omega \in \Omega_F$, a discrete DFRA life distribution that belongs to $G_{D,e}$.

$$(4.9) \qquad \overline{G}(k, \omega) = \begin{cases} \overline{F}(k) , & k \notin I_{2q}, & & 1 \leq q < m+1 \\[2ex] \overline{F}_2(k), & k \in I_{2q}, & \omega_{2q} = 1, & 1 \leq q < m+1, \\[2ex] \overline{F}_1(k), & k \in I_{2q}, & \omega_{2q} = 0, & 1 \leq q < m+1. \end{cases}$$

Note that for each $\omega \in \Omega_F$, $\overline{G}(\cdot, \omega) \in G_{D,e}$.

Finally, we prove Statement (4.1).

Theorem 4.1. Let F be a discrete DFRA life distribution that does not belong to $G_{D,e}$. Then for $k = 0, 1, \ldots$, $\overline{F}(k) = \int_{\Omega_F} \overline{G}(k, \omega) dP_F(\omega)$.

Proof. Let $k \in \{0, 1, \ldots\}$, and let I denote the indicator function. Then:

$$\int_{\Omega_F} \overline{G}(k, \omega) dP_F(\omega) = \overline{F}(k) I(k \notin \bigcup_{q=1}^{m} I_q)$$

$$+ \sum_{q=1}^{m} I(k \in I_{2q})\{\theta_{2q}\overline{F}_2(k) + (1 - \theta_{2q})\overline{F}_1(k)\}.$$

Consequently the desired result follows by (4.3). ||

Finally we note that, as is frequently the case, the representation is not unique.

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